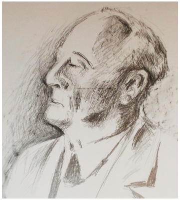
Functional Analysis

Bartosz Kwaśniewski

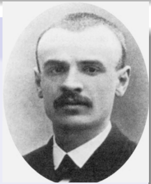
Faculty of Mathematics, University of Białystok



Lecture 12

Banach-Steinhaus Theorem

Thm. (Baire Theorem)



A countable union of nowhere dense sets in a complete metric space has empty interior

$$\left(\begin{array}{l} X \text{ - complete metric space} \\ A_n \subseteq X \text{ closed and } \operatorname{Int}(A_n) = \emptyset \end{array}\right) \implies \operatorname{Int}\left(\bigcup_{n=1}^{\infty} A_n\right) = \emptyset.$$

Ex. Let $X = \mathbb{Q} = \{q_1, q_2, \ldots\}$ with metric $d(x, y) = |x - y|$ and let $A_n = \{q_n\}$. Then A_n closed, $\operatorname{Int}(A_n) = \emptyset$, but $\bigcup_{n=1}^{\infty} A_n = \mathbb{Q}$. Hence $\operatorname{Int}(\bigcup_{n=1}^{\infty} A_n) = \operatorname{Int}(\mathbb{Q}) = \mathbb{Q} \neq \emptyset$. Why? (because $\mathbb{Q}$ is not complete!)

Rem. By the duality between open and closed sets, Baire's theorem can be equivalently formulated as follows:

A countable intersection of open dense sets in a complete metric space is a dense set

$$\left(\begin{array}{l} X \text{ - complete metric space} \\ U_n \subseteq X \text{ open and } \overline{U_n} = X \end{array}\right) \implies \overline{\bigcap_{n=1}^{\infty} U_n} = X.$$

Banach-Steinhaus Theorem

Let X be a Banach space and Y a normed space unormowana. For any familly $\{T_i\}_{i\in I} \subseteq B(X,Y)$ of bounded operators

$$\forall_{x\in X} \sup_{i\in I} \|T_i x\| < \infty \iff \sup_{i\in I} \|T_i\| < \infty. \qquad \overset{\Longleftarrow}{\text{easy}}$$

That is, $\forall_{x\in X}$ **the family** $\{T_i x\}_{i\in I}$ **is bounded in** Y ***(pointwise)*** $\iff$ **the family** $\{T_i\}_{i\in I}$ **is bounded in** $B(X,Y)$ ***(uniformly)***.

Proof: '$\Longrightarrow$' The sets $A_n := \{x \in X : \sup_{i\in I} \|T_i x\| \leqslant n\}$, $n \in \mathbb{N}$, are closed, because T_i are bounded. By assumption $X = \bigcup_{n=1}^{\infty} A_n$. By **Baire thm** $K(x_0, \varepsilon) \subseteq A_{n_0}$ for some $n_0 \in \mathbb{N}$, $x_0 \in X$ and $\varepsilon > 0$.
For $x \in X$, $\|x\| = 1$, and $i \in I$ we have

$$\|T_i x\| = \frac{2}{\varepsilon} \left\| T_i\left(\frac{\varepsilon}{2}x\right)\right\| = \frac{2}{\varepsilon} \left\| T_i\left(\left(x_0 + \frac{\varepsilon}{2}x\right) - x_0\right)\right\|$$

$$\leqslant \frac{2}{\varepsilon} \left\| T_i\left(x_0 + \frac{\varepsilon}{2}x\right)\right\| + \frac{2}{\varepsilon}\left\| T_i(x_0)\right\| \quad \left\{ \begin{array}{l} x_0 + \frac{\varepsilon}{2}x \in K(x_0,\varepsilon) \subseteq A_{n_0} \\ x_0 \in K(x_0,\varepsilon) \subseteq A_{n_0} \end{array} \right\}$$

$$\leqslant \frac{2}{\varepsilon} n_0 + \frac{2}{\varepsilon} n_0 = \frac{4}{\varepsilon} n_0. \quad \text{Hence } \sup_{i\in I} \|T_i\| \leqslant \frac{4}{\varepsilon} n_0 < \infty. \qquad \blacksquare$$

Cor. Pointwise limit of a sequence of bounded operators on a Banach space is a bounded operator. That is,

$$\begin{pmatrix} \{T_n\}_{n=1}^{\infty} \subseteq B(X,Y) \\ X \text{ is a Banach space} \\ \forall_{x \in X}\ \{T_n x\}_{n=1}^{\infty} \text{ convergent} \end{pmatrix} \implies \begin{pmatrix} T \in B(X,Y), \text{ where} \\ \forall_{x \in X}\ Tx := \lim_{n\to\infty} T_n x \end{pmatrix}$$

Proof: If $\{T_n x\}_{n=1}^{\infty}$ converges for every $x \in X$, then by putting $Tx := \lim_{n\to\infty} T_n x$ we obtain a linear operator, because the limit is a linear operation. Moreover, the convergence of the sequence $\{T_n x\}_{n=1}^{\infty}$ implies its boundedness, for every $x \in X$. Therefore, by the Banach–Steinhaus Theorem we have $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$.

Moreover

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leqslant \sup_{n\in\mathbb{N}} \|T_n x\| \leqslant \sup_{n\in\mathbb{N}} \|T_n\| \cdot \|x\|.$$

Hence T is bounded and $\|T\| \leqslant \sup_{n\in\mathbb{N}} \|T_n\| < \infty$. ■

Def. The **weak topology** on a normed space X is the weakest topology for which all functionals in X^* are continuous. The basis of this topology are sets of the form

$$U_{f_1,\dots,f_n,\varepsilon}(x) := \{y \in X : |f_i(y) - f_i(x)| < \varepsilon,\ 1 \leqslant i \leqslant n\},$$

where $x \in X$, $f_1, \dots, f_n \in X^*$, $\varepsilon > 0$.

Rem. If the sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ is **weakly convergent** (i.e. convergent in the weak topology) to $x_0 \in X$, then we write $x_n \xrightarrow{w} x_0$. We have

$$x_n \xrightarrow{w} x_0 \iff \forall_{f \in X^*}\ f(x_n) \longrightarrow f(x_0).$$

Hahn–Banach theorem implies that the limit of a weakly convergent sequence is uniquely determined - the weak topology satisfies the Hausdorff condition. 

Rem. The topology on X given by the norm is stronger than the weak topology (hence the name of the latter): $x_n \xrightarrow{\|\cdot\|} x_0 \implies x_n \xrightarrow{w} x_0$.

Ex. If $X = H$ is a Hilbert space, then by the Riesz-Fréchet every functional $f \in H^*$ is of the form $f(x) = \langle x, y \rangle$, for certain $y \in H$. Hence for every sequence $\{x_n\}_{n=1}^{\infty} \subseteq H$ we have

$$x_n \xrightarrow{w} x_0 \iff \forall_{y \in H}\ \langle x_n, y \rangle \longrightarrow \langle x_0, y \rangle.$$

For instance, consider an orthonornomal sequence $\{e_n\}_{n=1}^{\infty} \subseteq H$. Then

$$\|e_n - e_m\|^2 = \|e_n\|^2 + 2\,\mathrm{Re}\langle e_n, e_m \rangle + \|e_m\|^2 = 2, \qquad n \neq m.$$

Hence $\{e_n\}_{n=1}^{\infty}$ is not convergent in norm. But it is weakly convergent:

$$e_n \xrightarrow{w} 0.$$

Indeed, for any $y \in H$, Bessel inequality gives $\sum_{n=1}^{\infty} |\langle e_i, y \rangle|^2 \leqslant \|y\|^2$. Since the series $\sum_{n=1}^{\infty} |\langle e_i, y \rangle|^2$ converges, we get $\langle e_i, y \rangle \to 0 = \langle 0, y \rangle$. Hence $e_n \xrightarrow{w} 0$.

The norm is not weakly convergent, as $\|0\| = 0 < 1 = \liminf_{n \to \infty} \|e_n\|$.

Thm1. Weak topology=norm topology $\iff \dim(X) < \infty$.

Thm2. X is reflexive $\iff \{x \in X : \|x\| \leqslant 1\}$ weakly compact.

Prop. Every weakly convergent sequence is bounded, that is

$$x_n \xrightarrow{w} x_0 \implies \{x_n\}_{n=1}^{\infty} \text{ bounded in norm.}$$

Moreover, $\|x_0\| \leqslant \liminf_{n\to\infty} \|x_n\|$ *(the norm is weakly lower semicontinuous)*.

Proof: We may treat $x \in X \subseteq X^{**}$ as the functional $i(x)$ on X^*, where $i(x)(f) = f(x)$. Then $\|i(x)\| = \|x\|$ and

$$x_n \xrightarrow{w} x_0 \iff \forall_{f\in X^*}\ i(x_n)(f) \longrightarrow i(x_0)(f).$$

That is, the weak convergence of the sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ is equivalent to the pointwise convergence of the sequence of linear functionals $\{i(x_n)\}_{n=1}^{\infty}$. Thus if $x_n \xrightarrow{w} x_0$, then by **Banach-Steinhaus thm** (see **Cor**), the sequence $\{i(x_n)\}_{n=1}^{\infty} \subseteq X^{**}$ is bounded.
By **Hahn-Banach thm** there is $f \in X^*$ such that $\|f\| = 1$ and $f(x_0) = \|x_0\|$. Hence, by the weak convergence,

$$\|x_0\| = f(x_0) = \lim_{n\to\infty} f(x_n) = \liminf_{n\to\infty} f(x_n)$$

$$\leqslant \liminf_{n\to\infty} \|f\|\|x_n\| = \liminf_{n\to\infty} \|x_n\|. \qquad \blacksquare$$